(b) Energy - Time uncertainty relation

- $\Delta t \, \Delta E \simeq \hbar$ is not about time uncertainty.

  : Remember, time is just a parameter!

- def. Correlation amplitude

$$C(t) \equiv \langle \alpha | \alpha ; t \rangle \qquad // \; t_0 = 0$$

: Resemblance between the state kets at different times.

For $|\alpha\rangle = \sum_n c_n |n\rangle$ // $|n\rangle$: energy eigenkets.

$$C(t) = \langle \alpha | \mathcal{U}(t) | \alpha \rangle$$

$$= \sum_n c_n^* \langle n | \cdot \mathcal{U}(t) \cdot \sum_{n'} c_{n'} |n'\rangle$$

$$= \sum_n |c_n|^2 \exp\left[- \frac{i E_n t}{\hbar} \right]$$

at $t=0$, $C(t)$; as t increases, $C(t)$ decreases if E_n is random.

(?)

(There are a lot deeper theories ..)

- If we consider a large system with a quasi-continuous spectrum,

$$\sum_n \rightarrow \int dE \, \underbrace{\rho(E)}_{\text{density of states}} \; , \quad c_n \rightarrow g(E_n)$$

$$\Rightarrow \quad C(t) = \int dE \, |g(E)|^2 \rho(E) \exp\left[\frac{-iEt}{\hbar} \right]$$

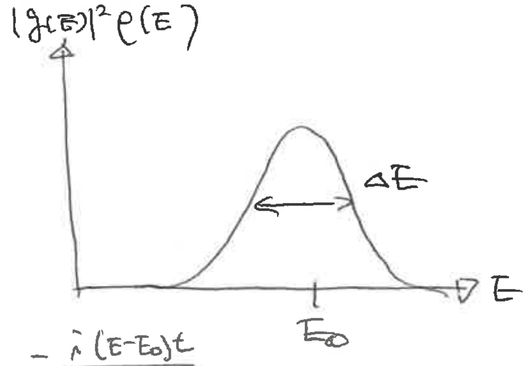
// normalization condition $\int dE \, |g(E)|^2 \rho(E) = 1$.

- If Energy is well defined,

$$E_0 = \langle H \rangle = \sum_n |c_n|^2 E_n \quad : \text{ time-indep.}$$

$$\rightarrow \int dE\, |g(E)|^2 \rho(E)\, E = E_0$$

meaning that $|g(E)|^2\rho(E)$ is peaked at $E = E_0$!



- Coming back to $C(t)$,

$$C(t) = \int dE\, |g(E)|^2 \rho(E)\, e^{-\frac{iEt}{\hbar}}$$

$$= e^{-\frac{iE_0 t}{\hbar}} \int dE\, |g(E)|^2 \rho(E)\, e^{-\frac{i(E-E_0)t}{\hbar}}$$

- $e^{-\frac{i(E-E_0)t}{\hbar}} \rightarrow$ when $\frac{(E-E_0)t}{\hbar} \ll 1$, (short time) "$\int dE \cdots e^{-i(E-E_0)t/\hbar} \rightarrow$ finite"

$\rightarrow$ when $t \gg \frac{\hbar}{\Delta E}$ $\quad \| \; |E-E_0| \lesssim \Delta E$

$$\int dE \cdots e^{-i(E-E_0)t/\hbar} \rightarrow 0 .$$

random phase !

$\Rightarrow$ "characteristic" time

$t \simeq \frac{\hbar}{\Delta E}$. (above which it loses the initial state !)

$\Rightarrow$ time - energy uncertainty relation

$$\boxed{\Delta t\, \Delta E \simeq \hbar}$$

(It has nothing to do with incompatible observables.)

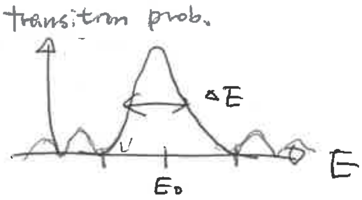
※ Δt : the time scale to retain the information of the original state,

ΔE : the relevant energy spread in the system.

* Another interpretation of $\Delta t \Delta E \sim \hbar$ in the perturbation theory.

Δt : duration of "drive" (ex. measurement time in spectroscopy.)

ΔE : spectral width of the transition obtained in exp.
(≅ uncertainty). i.e.



Now, this is not given but measured."

2.2 Schrödinger vs. Heisenberg picture.

(1) Two interpretations of the unitary transformation.

Consider

$$\langle\beta| X |\alpha\rangle \xrightarrow{U} \langle\beta| U^\dagger X U |\alpha\rangle$$

- interpretation 1.

$|\alpha\rangle \longrightarrow U|\alpha\rangle$: the state is changed.

$X \longrightarrow X$: the operator is unchanged.

- interpretation 2.

$|\alpha\rangle \longrightarrow |\alpha\rangle$: the state is unchanged.

$X \longrightarrow U^\dagger X U$: the operator is changed.

In fact, the interpretation 2. is more classical-mechanics friendly!

In the classical mechanics,

$$X \to X + \delta X, \quad L \to L + \delta L, \quad \cdots$$

$\Rightarrow$ [QM ver.2] $\hat{X} \to \hat{X} + \delta X$, $\hat{L} \to \hat{L} + \delta L$, ...

ex. $\mathcal{J}(\delta x)$: infinitesimal position translation.

$$\tilde{x} \to \left(1 + \frac{i\tilde{p}\delta x}{\hbar}\right) \tilde{x} \left(1 - \frac{i\tilde{p}\delta x}{\hbar}\right)$$

$$= \tilde{x} + \frac{i}{\hbar}[\hat{p}\delta x, \tilde{x}]$$

$$= \tilde{x} + \delta x$$

$\Rightarrow$ measurement

$$\langle \tilde{x} \rangle = \langle \tilde{x} \rangle + \langle \delta x \rangle$$

$\mathcal{J}(\delta x)|x\rangle$

"the same" result!

$\mathcal{J}^\dagger \tilde{x} \mathcal{J}$

Interpretation 1 $\to$ "Schrödinger picture"

The state ket is evolving.

Interpretation 2 $\to$ "Heisenberg picture"

The operator is evolving.

(2) State kets and Observables in the two pictures

$$A^{(H)}(t) = \mathcal{U}^\dagger(t) A^{(S)} \mathcal{U}(t) \qquad \Bigg\| \qquad A^{(H)}(0) = A^{(S)}$$

Heisenberg   Schrödinger

$$|\alpha, t_0 = 0; t\rangle_H = |\alpha, t_0 = 0\rangle$$

$$|\alpha, t_0 = 0; t\rangle_S = \mathcal{U}(t)|\alpha, t_0 = 0\rangle.$$

$\langle A \rangle$: unchanged.

(3) Heisenberg Equation of Motion

- We need an equation for the time-evolution of an "operator".

$$\frac{dA^{(H)}}{dt} = \frac{d}{dt}\left(U^{+}A^{(S)}U\right) = \frac{\partial U^{+}}{\partial t}A^{(S)}U + U^{+}A^{(S)}\frac{\partial U}{\partial t}$$

$$= -\frac{1}{i\hbar}U^{+}H\underbrace{\,}_{UU^{+}}A^{(S)}U + U^{+}A^{(S)}\underbrace{\,}_{UU^{+}}\cdot\frac{1}{i\hbar}HU$$

$$= \frac{1}{i\hbar}\left[-U^{+}HU\underbrace{A^{(H)}}_{\equiv A(t)} + A^{(H)}U^{+}HU\right]$$

$$= \frac{1}{i\hbar}\left[A^{(H)}, \underbrace{U^{+}HU}_{=H}\right] \quad \left(\because [U, H] = 0.\right)$$

$$\Rightarrow \boxed{\frac{dA^{(H)}}{dt} = \frac{1}{i\hbar}\left[A^{(H)}, H\right]} + \left(\frac{\partial A^{H}}{\partial t}\right)$$

Heisenberg EOM

↳ when A^{H} has an explicit time-dependence.

> * Notation note.
>
> Often, we write $\begin{bmatrix} A^{(H)} \equiv A(t) \\ A^{(S)} = A \end{bmatrix}$.

Classical - Quantum correspondence

$$\frac{[\ ,\ ]^{QM}}{i\hbar} \longleftrightarrow [\ ,\ ]_{classical}$$

$$\frac{dA}{dt} = [A, H]_{classical}$$

↑ Poisson Bracket.

(4) Free particles ; Ehrenfest's Theorem.

$$H = \frac{\tilde{\vec{p}}^{\,2}}{2m} = \frac{1}{2m}\left(\tilde{p}_x^2 + \tilde{p}_y^2 + \tilde{p}_z^2\right).$$

Heisenberg EOM //

// NOTE: All operators are in the Heisenberg picture.

① $$\frac{d\tilde{p}_i}{dt} = \frac{1}{i\hbar}\left[\tilde{p}_i, H\right] = 0 \quad : \text{ conserved !}$$

② $$\frac{d\tilde{x}_i}{dt} = \frac{1}{i\hbar}\left[\tilde{x}_i, H\right] = \frac{1}{i\hbar}\,\frac{1}{2m}\, i\hbar \frac{\partial}{\partial \tilde{p}_i}\left(\sum_{j=1}^{3} \tilde{p}_j^2\right).$$

$$= \frac{\tilde{p}_i}{m} = \frac{\tilde{p}_i(0)}{m} \quad (\text{invariant}).$$

|| $[\tilde{x}_i, F(\tilde{\vec{p}})] = i\hbar \dfrac{\partial F}{\partial \tilde{p}_i}$

$[\tilde{p}_i, G(\tilde{\vec{x}})] = -i\hbar \dfrac{\partial G}{\partial \tilde{x}_i}$

$$\Rightarrow \quad \tilde{x}_i(t) = \tilde{x}_i(0) + \frac{\tilde{p}_i(0)}{m}\cdot t. \quad //$$

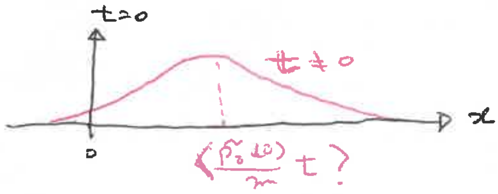
It looks like a classical dynamics, BUT

note that ...

$$[\tilde{x}_i(0), \tilde{x}_j(0)] = 0.$$

$$[\tilde{x}_i(t), \tilde{x}_j(0)] = \left[\frac{\tilde{p}_i(0)}{m}t, x_j(0)\right] = -\frac{i\hbar t}{m} \neq 0 \quad //$$

$\therefore$ $\hat{x}$ operator spreads over distance in time.



uncertainty relation

$$\langle (\Delta \tilde{x}_i)^2\rangle_t \langle (\Delta \tilde{x}_i)^2\rangle_0 \geq \frac{\hbar^2 t^2}{4m^2}.$$

Now, adding a potential $V(\vec{x})$,

$$H = \frac{\vec{p}^{\,2}}{2m} + V(\vec{x})$$

EOM :
$$\frac{dp_i}{dt} = \frac{1}{i\hbar}\left[p_i, V(\vec{x})\right] = -\frac{\partial}{\partial x_i} V(\vec{x})$$

$$\frac{dx_i}{dt} = \frac{p_i}{m}.$$

also,
$$\frac{d^2 x_i}{dt^2} = \frac{1}{i\hbar}\left[\frac{p_i}{m}, H\right] = \frac{1}{m}\frac{dp_i}{dt}.$$

$$\therefore\; m\frac{d^2\vec{x}}{dt} = -\nabla V(\vec{x}) \quad :$$ QM ver. Newton's second law!

for expectation values, $\because$ $|\alpha\rangle$ is t-indep. in the Heisenberg picture!

$$m\frac{d^2}{dt^2}\langle \vec{x} \rangle = \frac{d\langle \vec{p} \rangle}{dt} = -\langle \nabla V(\vec{x}) \rangle.$$

← NOTE THAT it's not $V(\langle x \rangle)$.

"Ehrenfest theorem."

(The center of a wave packet moves like a classical particle.)

Valid only in the Heisenberg picture.

independent of the pictures.

(5) Base Kets and Transition amplitudes

	Schrödinger	Heisenberg
State ket	Moving.	Stationary
Observable	Stationary	Moving.
Base ket	Stationary	Moving oppositely.

- Base kets in the Schrödinger picture.

  Operator : time - independent

  $\Rightarrow \quad A|a\rangle = a|a\rangle$

- In the Heisenberg picture

  $A^{(H)}(t) = U^{\dagger} A U$

  Thus. $\underset{U^{\dagger}}{\frown} A \underset{U U^{\dagger}}{\frown} |a\rangle = a \underset{U^{\dagger}}{\frown} |a\rangle$ becomes

  $\Rightarrow \quad A^{(H)}(t) \left(U^{\dagger}|a\rangle \right) = a \left(U^{\dagger}|a\rangle \right).$

  $\Rightarrow \quad \underbrace{|a,t\rangle_H \equiv U^{\dagger}|a\rangle}_{\text{time-dependent}}.$: Base kets in the Heisenberg picture.

- Time-evolution of $|a,t\rangle_H$.

  $i\hbar \frac{\partial}{\partial t}|a,t\rangle_H = i\hbar \frac{\partial}{\partial t} U^{\dagger}|a\rangle = -H U^{\dagger}|a\rangle$

  $\Rightarrow \quad \underbrace{i\hbar \frac{\partial}{\partial t}|a,t\rangle_H = -H|a,t\rangle_H}$

  moving in an opposite way!

- expansion coefficient $C_a(t)$

  - Schrödinger picture :

    $|\alpha,t\rangle = \sum_a C_a(t)|a\rangle \Rightarrow C_a(t) = \underset{\text{base bra}}{\langle a|} \cdot \underbrace{U|\alpha,t=0\rangle}_{\text{state ket}}$

    $\|$

  - Heisenberg picture :

    $|\alpha\rangle = \sum_a C_a(t)|a,t\rangle_H \Rightarrow C_a(t) = \underbrace{\langle a|U}_{\text{base bra}} \cdot \underset{\text{state ket}}{|\alpha\rangle}$

  $\Rightarrow$ Transition probability

* The temporal Heisenberg inequality

- Ehrenfest theorem:

$$\frac{d}{dt}\langle A\rangle_\varphi = \frac{1}{i\hbar}\langle [A,H]\rangle_\varphi$$

$\| \ \langle\cdot\rangle_\varphi = \langle\varphi(t)|\cdot|\varphi(t)\rangle$

- uncertainty relation: $\langle(\Delta A)^2\rangle_\varphi \langle(\Delta B)^2\rangle_\varphi \geq \frac{1}{4}|\langle[A,B]\rangle_\varphi|^2$

$\langle(\Delta A)^2\rangle_\varphi \equiv (\Delta_\varphi A)^2$.

Let's put H into B!

$$\rightarrow \quad \Delta_\varphi H \, \Delta_\varphi A \geq \frac{1}{2}|\langle[A,H]\rangle_\varphi| = \frac{1}{2}\hbar\left|\frac{d}{dt}\langle A\rangle_\varphi\right|$$

If we define the time $\tau_\varphi(A)$ as

$$\frac{1}{\tau_\varphi(A)} \equiv \left|\frac{d\langle A\rangle_\varphi}{dt}\right|\frac{1}{\Delta_\varphi A},$$

then τ_φ = characteristic time for the expectation value of A to change by $\Delta_\varphi A$.

$$\Rightarrow \quad \Delta_\varphi H \, \tau_\varphi(A) \geq \frac{1}{2}\hbar \quad \Rightarrow \quad \text{"}\Delta E \Delta t \gtrsim \frac{1}{2}\hbar\text{"}$$

Energy spread — characteristic evolution time.

2.3 Simple Harmonic oscillator

(1) Energy eigenkets. (Dirac's operator method)

$$H = \frac{\tilde{p}^2}{2m} + \frac{1}{2}m\omega^2\tilde{x}^2 = \hbar\omega\left(\tilde{a}^\dagger\tilde{a} + \frac{1}{2}\right)$$

$$\equiv \hbar\omega\left(\tilde{N} + \frac{1}{2}\right).$$

def. annihilation operator

$$\tilde{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\tilde{x} + i\frac{\tilde{p}}{m\omega}\right)$$

creation operator

$$\tilde{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\tilde{x} - i\frac{\tilde{p}}{m\omega}\right)$$

$$\Rightarrow \quad \tilde{x} = \frac{x_0}{\sqrt{2}}(\tilde{a} + \tilde{a}^\dagger), \qquad \tilde{p} = i\frac{\hbar}{\sqrt{2}x_0}(-\tilde{a} + \tilde{a}^\dagger)$$

$\| \ x_0 = \sqrt{\frac{\hbar}{m\omega}}$

$$\tilde{N} = \tilde{a}^\dagger\tilde{a}$$

$\rightarrow$ Commutation relation $[\tilde{a}, \tilde{a}^\dagger] = 1$